Chapter 5

5.1 Relations

Definition 16 *Let S be a set. If x and y are elements of the set S, then the pair (x, y) is called an* **ordered pair** *of elements in S. For the ordered pair, (x, y), x is called the* **first elment** *and y is called the* **second element** *of the order pair. Two ordered pairs, (x, y) and (u, v) are* **equal** *means $x = u$ and $y = v$. The collection of ordered pairs with all elements of the set S as first and second elements is called the* **Cartesian product of S with itself**, *denoted by $S \times S$.*

Definition 17 *Let S and T be sets. S is a* **subset** *of T means every element of S is an element of T. S is* **equal** *to T means S is a subset of T and T is a subset of S.*

Definition 18 *Let S be a set. A* **relation** *R on S is any subset of $S \times S$. The* **domain** *of the relation R on S is the set of all first elements in R and the* **range** *of the relation R on S is the set of all second elements in R.*

Definition 19 *Let S be a set and R be a relation on S. R is called an* **equivalence relation** *on S means each of the following is true:*

1. *For every x in S, (x, x) is in R. This is called the* **reflexive property**.
2. *If (x, y) is in R, then (y, x) is in R. This is called the* **symmetric property**.
3. *If each of (x, y) and (y, z) is in R, then (x, z) is in R. This is called the* **transitive property**.

Remark 1 *For simplicity, given a set S and an equivalence relation R on S, we shall sometimes use the notation xRy to mean (x, y) is in R.*

Example 1 *Let N denote the set of natural numbers and let $S = N \times N$. If each of (x, y) and (u, v) is an element of S, we define the relation R on S to be the subset of $S \times S$ such that*

$$R = \{((x, y), (u, v)) : x + v = y + u\}.$$

Show R is an equivalence relation on S.

Example 2 *Let N denote the set of natural numbers and let $S = N \times N$. If each of (x, y) and (u, v) is an element of S, we define the relation T on S to be the subset of $S \times S$ such that*

$$T = \{((x, y), (u, v)) : x \cdot v = y \cdot u\}.$$

Show T is an equivalence relation on S.

Definition 20 *Let S be a set, R be an equivalence relation on S, and also let s be an element in S. The collection of all elements y in S such that (s, y) is in R is called the* **equivalence class of** *s, denoted by s^R.*

Example 3 *Using Example* **??** *before, the equivalence classes of each element (a, b) in S contain abitrarily many elements, namely*

$$(a, b)^R = \{(x, y) : a + y = b + x\}.$$

List six elements of $(7, 8)^R$.

Example 4 *Using Example 2, list six elements of $(7, 8)^T$.*

Theorem 21 *Let S denote a set and R an equivalence relation S. If each of s and t is an element of S such that sRt, then $s^R = t^R$.*

Theorem 22 *Let S denote a set and R an equivalence relation on S. If each of s and t are elements of S, then either $s^R = t^R$ or s^R and t^R have no common elements.*

5.2 Integers

Definition 21 *(From Example1 of the previous section) Let N denote the set of natural numbers and let $S = N \times N$. If each of (x, y) and (u, v) is an element of S, define the relation R on S to be the subset of $S \times S$ such that*

$$R = \{((x, y), (u, v)) : x + v = y + u\}$$

We'll say $(x, y)R(u, v)$ to mean the pair $((x, y), (u, v))$ is in R.

Theorem 23 *The relation R on S from Definition* **??** *is an equivalence relation on S.*

Theorem 24 *If R is the equivalence relation on S from Definition* **??***, then for each (a, b) and (c, d) in S,*

$$(a, b)^R = (c, d)^R \text{ if and only if } a + d = b + c.$$

Definition 22 *Let R be the equivalence relation on S from Definition* **??***. Let $Z = \{(a, b)^R : (a, b) \text{ is in } S\}$. Each $(a, b)^R$ is called an integer and Z is called the set of integers.*

Theorem 25 *Two integers, $(a, b)^R$ and $(c, d)^R$, are equal if and only if $a + d = b + c$.*

Definition 23 *Let* $(a,b)^R$ *and* $(c,d)^R$ *denote integers and define the sum of* $(a,b)^R$ *and* $(c,d)^R$*, denoted by* $(a,b)^R + (c,d)^R$*, to be the integer* $(a+c, b+d)^R$*.*

Theorem 26 *If each of* $(a,b)^R$ *and* $(c,d)^R$ *is an integer, then*

$$(a,b)^R + (c,d)^R = (c,d)^R + (a,b)^R.$$

Theorem 27 *For integers,* $(a,b)^R$*,* $(c,d)^R$*, and* $(e,f)^R$*,*

$$(a,b)^R + ((c,d)^R + (e,f)^R) = ((a,b)^R + (c,d)^R) + (e,f)^R.$$

Remark 2 *The integer* $(a,a)^R$ *has an unusual characteristic. It is given in the next theorem. We shall call such a number an* **additive identity***.*

Theorem 28 *If* $(c,d)^R$ *is any integer, then*

$$(c,d)^R + (a,a)^R = (c,d)^R.$$

Theorem 29 *The additive identity for the set of integers is unique and has the form that if* a *is a natural number, then the additive identity is* $(a,a)^R$*.*

Theorem 30 *If* $(a,b)^R$*,* $(c,d)^R$*, and* $(e,f)^R$ *are integers, and if* $(a,b)^R + (e,f)^R = (c,d)^R + (e,f)^R$ *then* $(a,b)^R = (c,d)^R$*.*

Theorem 31 *If* $(c,d)^R$ *is an integer, then there exists one and only one integer* $(e,f)^R$ *such that*

$$(c,d)^R + (e,f)^R = (a,a)^R.$$

Definition 24 *Given an integer* $(c,d)^R$*, the integer* $(e,f)^R$ *from the above theorem is called the* **additive inverse** *of* $(c,d)^R$ *and is denoted by* $-(c,d)^R$*.*

5.3 Multiplication – Integers

Definition 25 *If* $(a,b)^R$ *and* $(c,d)^R$ *are integers, the product of* $(a,b)^R$ *and* $(c,d)^R$*, denoted by* $(a,b)^R \cdot (c,d)^R$*, is the integer given by* $(ac+bd, ad+bc)^R$*.*

Theorem 32 *If* $(a,b)^R$ *and* $(c,d)^R$ *are integers, then*

$$(a,b)^R \cdot (c,d)^R = (c,d)^R \cdot (a,b)^R.$$

Theorem 33 *If* $(a,b)^R$*,* $(c,d)^R$*, and* $(e,f)^R$ *are integers, then*

$$((a,b)^R \cdot (c,d)^R) \cdot (e,f)^R = (a,b)^R \cdot ((c,d)^R \cdot (e,f)^R).$$

Theorem 34 *If* $(c,d)^R$ *is any integer, then*

$$(c,d)^R \cdot (a,a)^R = (a,a)^R.$$

Definition 26 *The integer* $(a+1,a)^R$ *has an important property with respect to multiplication which is given in the next theorem.* $(a+1,a)^R$ *is called a* **multiplicative identity**.

Theorem 35 *If* $(c,d)^R$ *is any integer, then*

$$(a+1,a)^R \cdot (c,d)^R = (c,d)^R.$$

Theorem 36 *The multiplicative identity for the set of integers is unique.*

Theorem 37 *For integers* $(a,b)^R$, $(c,d)^R$, *and* $(e,f)^R$,

$$(a,b)^R \cdot ((c,d)^R + (e,f)^R) = (a,b)^R \cdot (c,d)^R + (a,b)^R \cdot (e,f)^R.$$

5.4 Some New Notation

New notation. Given the integer $(a, b)^R$, we know that a, b are natural numbers so, by trichotomy, exactly one of the following is true:

1. $a = b$.
2. $a < b$.
3. $a > b$.

We shall simplify our notation for integers $(a, b)^R$ in the following way:

1. The symbol 0 denotes the integer $(a, b)^R$ when $a = b$.
2. The symbol $+p$ is used to denote the integer $(a, b)^R$ when $a > b$ and p is the natural number such that $a = b + p$. Such integers are called **positive integers**.
3. The symbol $-q$ is used to denote the integer $(a, b)^R$ when $a < b$ and q is the natural number such that $a + q = b$. These integers are called **negative integers**.

5.4.1 Exercises

1. Show that $(5, 3)^R + (2, 5)^R = (5, 6)^R$.
2. Show that $(+2) + (-3) = (-1)$.
3. If n is a natural number, then $(n, 2n)^R = -n$.
4. If n is a natural number, then $-(n, 2n)^R = +n$.
5. Prove that the product of two positive integers is a positive integer.
6. Prove that the sum of two negative integers is a negative integer.
7. Prove that the sum of two positive integers is a positive integer.
8. Prove that the product of two negative integers is a positive integer.
9. Prove that the product of a negative integer and a positive integer is a negative integer.
10. Suppose the sum of two natural numbers a, b is the natural number c. Prove that
$$(+a) + (+b) = +c.$$
11. Suppose the product of two natural numbers a, b is the natural number c, prove that
$$(+a)(+b) = +c.$$
12. Given natural numbers a, b, prove that:
    (a) $-((+a) + (+b)) = (-a) + (-b)$.
    (b) $(+a) + (-b) = -((+b) + (-a))$.

5.5 Order – Integers

Definition 27 *Given integers a and b, we say that a is less than b, denoted by $a < b$, provided there is a positive integer p such that $a + p = b$. Also $a > b$ if and only if $b < a$; $a \leq b$ if and only if $a < b$ or $a = b$; and $a \geq b$ if and only if $a > b$ or $a = b$.*

Theorem 38 *An integer a is positive if and only if $a > 0$.*

Theorem 39 *An integer a is negative if and only if $a < 0$.*

Theorem 40 *If a and b are integers, then exactly one of the following holds:*

1. $a = b$.
2. $a < b$.
3. $a > b$.

Theorem 41 *Let a, b, and c denote integers. If $a < b$ and $b < c$, then $a < c$.*

Remark 3 *Note that since the integers have both trichotomy and transitive properties, the set of integers is an* **ordered set**.

Theorem 42 *If a, b, and c are integers and $a < b$, then $a + c < b + c$.*

Theorem 43 *If a, b, and c are integers and $a + c < b + c$, then $a < b$.*

Theorem 44 *If a, b, and c are integers and $a < b$ and $c > 0$, then $ac < bc$.*

Theorem 45 *If a, b, and c are integers and $a < b$ and $c < 0$, then $ac > bc$.*

Theorem 46 *If a, b, and c are integers, c is not 0 and $ac = bc$, then $a = b$.*

5.5.1 Exercises

1. If a and b are integers, show that $ab = 0$ if and only if $a = 0$ or $b = 0$.
2. Show there is no integer a such that $2a = 1$.
3. Show that for integers a and b, $ab = 1$ if and only if $a = b = 1$ or $a = b = -1$.
4. Show that if a, b, c are integers such that $ac < bc$ and $c > 0$, then $a < b$.
5. Show that if a, b, c are integers such that $ac < bc$ and $c < 0$, then $a > b$.

5.6 A New Relation

Theorem 47 *Let* Z *be the set of integers and let* $S = Z \times (Z - \{0\})$*. Note that if a pair* (a, b) *is in* S*, then* b *is not* 0*. Prove that the relation* F *on* S *given by*

$$F = \{((a, b), (c, d)) : ad = bc\}$$

is an equivalence relation on S*.*

Example 5 *Using the relation* F *above, list six elements in each of the following equivalence class:* $(2, 3)^F$*,* $(5, 1)^F$*,* $(-7, 14)^F$*,* $(-6, -2)^F$*, and* $(0, 10)^F$*.*